# Additional Exercises for 'Topics in Geometry'.

## Connections and Curvature.

*Exercise* 1. Let V be a vector space over $\mathbf{R}$ of dimension $n$. We consider multilinear maps

$$R : \mathrm{V} \times \mathrm{V} \times \mathrm{V} \times \mathrm{V} \to \mathbf{R}$$

which are 'algebraic curvature tensors' in the sense that

$$R(x, y, z, w) = -R(y, x, z, w) = -R(x, y, w, z)$$
$$R(x, y, z, w) + R(x, z, w, y) + R(x, w, y, z) = 0.$$

for all $x, y, z, w \in \mathrm{V}$.
(i) Any such $R$ satisfies $R(x, y, z, w) = R(z, w, x, z)$, and if $R(x, y, x, y) = 0$ for all $x, y \in \mathrm{V}$, then $R = 0$.
(ii) The dimension of the space of algebraic curvature tensors is $n^2(n^2 - 1)/12$.
(iii) Assume now that V carries an inner product $(\cdot, \cdot)$. The multilinear map $Q$ given by $Q(x, y, z, w) = (x, z)(y, w) - (y, z)(w, x)$ is an algebraic curvature tensor. Suppose that $R$ is an algebraic curvature tensor, and define $K(P) = R(p_1, p_2, p_1, p_2)$ for any plane $P \subset \mathrm{V}$ with orthonormal basis $(p_1, p_2)$. If there exists a constant $K$ with $K = K(P)$ for all $P$, then $R = KQ$. [Show that for any basis $(x, y)$ of $P$ we have $K(P) = R(x, y, x, y)/Q(x, y, x, y)$ and use (i).]
(iv) Use (iii) to deduce an expression for the Riemann curvature tensor of $\mathrm{S}^n$.

*Exercise* 2. Let $\gamma$ be a loop in $\mathrm{S}^2$ with $p = \gamma(0) = \gamma(1)$. The parallel transport map $P_\gamma$ is in $\mathrm{SO}(\mathrm{TS}^2_p)$, and hence corresponds to an angle $\theta \in \mathbf{R}/2\pi\mathbf{Z}$ ('holonomy angle').
(i) Compute the holonomy angle in the case where $\gamma$ is circle of latitude.
(ii) More generally, derive an expression for the holonomy angle for any simple closed loop $\gamma$. [Use the Gauss-Bonnet formula.]

## Chern Classes.

*Exercise* 3. Suppose that the tangent bundle of real projective $n$-space is trivial. Show that $n + 1$ is a power of 2. [Compute the total Stiefel-Whitney class of the tangent bundle].

*Exercise* 4. Let M be a compact oriented smooth manifold of dimension $m$.
(i) Let $a_1, \dots, a_r$ be a basis of $\mathrm{H}^*(\mathrm{M}, \mathbf{Q})$, and $b_1, \dots, b_r$ be the dual[1] basis. Then the Poincare dual of the diagonal $\Delta \subset \mathrm{M} \times \mathrm{M}$ can be expressed as

$$\delta = \sum_{k=1}^{r} (-1)^{|a_k|} a_k \times b_k,$$

where $\times$ is the cross product in cohomology. [Show that both sides have the same intersection form with $b_i \times a_j$ for all $i, j$ with $|b_i| + |a_j| = m$. ]
(ii) Use (i) to deduce the equality

$$\int_{\mathrm{M}} e(\mathrm{T_M}) = \sum_{i=0}^{m} (-1)^i \dim_{\mathbf{Q}} \mathrm{H}^i(\mathrm{M}, \mathbf{Q}).$$

How is this related to the Gauss-Bonnet theorem and the Poincare-Hopf theorem? [Let $\Delta : \mathrm{M} \to \mathrm{M} \times \mathrm{M}$ be the diagonal map. Then $\mathrm{T_M} \simeq \Delta^* \mathrm{N}_{\Delta/\mathrm{M}\times\mathrm{M}}$ implies $e(\mathrm{T_M}) = \Delta^* \delta$.]

*Exercise* 5. (i) The tangent bundle of a Lie group G is trivial, in particular orientable. Use exercise 5 to conclude that $\chi(\mathrm{G}) = 0$ if G is compact.
(ii) Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. If G is not commutative, then $\mathrm{H}^3(\mathrm{G}; \mathbf{R}) \neq 0$. [Let $\langle -, - \rangle$ be a bi-invariant Riemannian metric on G. The multilinear map $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ given by $(x, y, z) \mapsto \langle [x, y], z \rangle$ induces a bi-invariant 3-form $\theta$ on G. Prove that $\theta$ is closed but not exact (if it were, we would have $\theta = 0$).]
(iii) For which $n$ does $\mathrm{S}^n$ admit a Lie group structure? [A commutative compact connected Lie group must be a torus (the exponential map is a surjective morphism of Lie groups).]
(iv) Show that the tangent bundle of $\mathrm{S}^7$ is trivial. [Use the octions to define a trivialisation.]

*Exercise* 6. Assume there exists a polynomial $\mathrm{Td}(\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3) = \alpha \mathrm{T}_1^3 + \beta \mathrm{T}_1 \mathrm{T}_2 + \gamma \mathrm{T}_3$ such that for every smooth projective 3-fold X we have

$$\chi(\mathcal{O}_{\mathrm{X}}) = \int_{\mathrm{X}} \mathrm{Td}(c_1, c_2, c_3)$$

Show that $\alpha = \gamma = 0$ and $\beta = 1/24$. [Consider $\mathrm{X} = \mathbf{P}^3, \mathbf{P}^2 \times \mathbf{P}^1, \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ to get a system of three equations in $\alpha, \beta, \gamma$.]

*Exercise* 7. (i) Let $\mathrm{C} \subset \mathrm{S}$ be a smooth curve in a smooth projective surface S. Prove the formula

$$\int_{\mathrm{C}} c_1(\mathrm{N}_{\mathrm{C/S}}) = \int_{\mathrm{S}} \mathrm{D}[\mathrm{C}] c_1(\mathrm{S}) - \chi(\mathrm{C}),$$

[1] With respect to the intersection form.

where D[C] is the Poincaré dual of [C].
(ii) Use (i) to deduce the degree-genus formula for $\mathrm{C} \subset \mathrm{S} = \mathbf{P}^2$.

*Exercise* 8. (i) Show that there exists an exact sequence (Euler sequence)

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus(n+1)} \to \mathrm{T}_{\mathbf{P}^n} \to 0.$$

How is it related to the tautological exact sequence? Compute $c(\mathrm{T}_{\mathbf{P}^n})$.
(ii) Compute the Euler characteristic

$$\chi(\mathrm{X}) = \int_{\mathrm{X}} c_{n-1}(\mathrm{T}_{\mathrm{X}})$$

of a smooth hypersurface $\mathrm{X} \subset \mathbf{P}^n$ of degree $d$. [Consider the short exact sequence $0 \to \mathrm{T}_{\mathrm{X}} \to \mathrm{T}_{\mathbf{P}^n}|\mathrm{X} \to \mathrm{N}_{\mathrm{X}/\mathbf{P}^n} \to 0$.]

*Exercise* 9. (i) Let $\mathcal{E}$ be a vector bundle of rank $e$, and $\mathcal{L}$ a line bundle. Prove

$$c_t(\mathcal{E} \otimes \mathcal{L}) = \sum_{j=0}^{e} c_j(\mathcal{E}) c_t(\mathcal{L})^{e-j} t^j.$$

(ii) Let $\mathbf{P}^n = \mathbf{P}(\mathrm{V})$ with tautological subbundle $\mathcal{S}$ and quotient bundle $\mathcal{Q}$, and $q$ (resp. $p$) denote the first (resp. second) projection of $\mathbf{P}^n \times \mathbf{P}^n$. Construct a morphism of bundles $q^*\mathcal{S} \to p^*\mathcal{Q}$ whose zero locus is exactly the diagonal $\Delta \subset \mathbf{P}^n \times \mathbf{P}^n$. [Use the tautological exact sequence; at a point $(x, y) \in \mathbf{P}^n \times \mathbf{P}^n$ corresponding to $\mathrm{L}_x, \mathrm{L}_y \subset \mathrm{V}$ the map on fibres should be $\mathrm{L}_x \to \mathrm{V}/\mathrm{L}_y$.]
(iii) Use (i) and (ii) to compute the class

$$\delta \in \mathrm{H}^n(\mathbf{P}^n \times \mathbf{P}^n; \mathbf{Z}) = \mathbf{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{n+1})$$

Poincaré dual to the diagonal.

*Exercise* 10 (Yau). (i) Consider the intersection $\mathrm{Z} \subset \mathbf{P}^3 \times \mathbf{P}^3$ of the hypersurfaces

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0,$$
$$y_0^3 + y_1^3 + y_2^3 + y_3^3 = 0,$$
$$x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0.$$

Compute $\chi(\mathrm{Z}) = -18$. [View Z as the zero scheme of a section of the vector bundle $\mathcal{E} = \mathcal{O}(3,0) \oplus \mathcal{O}(0,3) \oplus \mathcal{O}(1,1)$. Note that

$$\left( \frac{c(\mathrm{T}_{\mathbf{P}^3 \times \mathbf{P}^3})}{c(\mathcal{E})} \right)_3 c_3(\mathcal{E}) = -18\alpha^3\beta^3$$

in $\mathrm{H}^n(\mathbf{P}^n \times \mathbf{P}^n; \mathbf{Z}) = \mathbf{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^{n+1})$.]
(ii) Let $\zeta$ be a primitive third root of unity, and consider the automorphism $\sigma =$

$\sigma_1 \times \sigma_2$ of $\mathbf{P}^3 \times \mathbf{P}^3$ given by

$$\sigma_1([x_0 : x_1 : x_2 : x_3]) = [x_1, x_2, x_0, \zeta x_3],$$
$$\sigma_2([y_0 : y_1 : y_2 : y_3]) = [y_1, y_2, y_0, \zeta^2 y_3].$$

Show that the cyclic group $\Sigma$ generated by $\sigma$ acts freely on Z, and conclude that $\mathrm{X} = \mathrm{Z}/\Sigma$ has $\chi(\mathrm{X}) = -6$. [If a finite group G acts on a compact manifold M, then $\chi(\mathrm{M}^{\mathrm{G}}) = \frac{1}{\#\mathrm{G}} \sum_{g \in \mathrm{G}} \chi(\mathrm{M}^g)$.]

**Complex Manifolds.**

*Exercise* 11. (i) Let X be a smooth projective variety of dimension $d$. Show that $\mathrm{H}_{2k}(\mathrm{X}; \mathbf{C}) \neq 0$ for $k \leqslant d$. [Embed X into some projective space and consider the intersection of X with linear subspace.]
(ii) Which spheres $\mathrm{S}^n$ can be the underlying topological space of a smooth projective variety?

*Exercise* 12. (i) Show that if $\mathrm{S}^n$ admits an almost complex structure, then $\mathrm{S}^{n+1}$ is parallelisable. [Let $e_1, \dots, e_{n+2}$ be the standard basis of $\mathbf{R}^{n+2}$, view $\mathrm{S}^n$ as the unit sphere in $\mathbf{R}^{n+1} = \langle e_1, \dots, e_{n+1} \rangle$. Use the almost complex structure J on $\mathrm{S}^n$ to define for every $p \in \mathrm{S}^{n+1}$ a linear map $\sigma_p : \mathbf{R}^{n+1} \to \mathrm{TS}^{n+1}_p$ such that the vector bundle map $\sigma : \mathrm{S}^{n+1} \times \mathbf{R}^{n+1} \to \mathrm{TS}^{n+1}$, $(p, v) \mapsto (p, \sigma_p(v))$ is an isomorphism. Note that any $p \in \mathbf{R}^{n+2}$ can uniquely be written as $p = \alpha e_{n+2} + \beta s$ with $s \in \mathrm{S}^n$, $\alpha \in \mathbf{R}$, $\beta \geqslant 0$.]
(ii) View $\mathrm{S}^6$ as the purely imaginary octonions of norm one, and use octonionic multiplication to define an almost complex structure on $\mathrm{S}^6$. Compute the Nijenhuis tensor to show that it is not integrable.

*Exercise* 13 (Borel-Serre). Show that if a sphere $\mathrm{S}^{2n}$ admits an almost complex structure, then $n \leqslant 3$. [If $S^{2n}$ has an almost complex structure, then the tangent bundle T of $\mathrm{S}^{2n}$ is a complex vector bundle. Compute the Chern character of T to see that the top-dimensional part is $c_n(\mathrm{T})/(n-1)!$. Assume that $c_n(\mathrm{T})$ is divisible by $(n-1)!$ in integral cohomology (this is nontrivial), and use exercise 5 to conclude that 2 is divisible by $(n-1)!$.]

**Hodge Theory.**

*Exercise* 14. (i) Compute the Hodge numbers of $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$.
(ii) Compute the Chern and Hodge numbers of $\mathbf{P}^3$ and a quadric threefold.

*Exercise* 15. Let X be a compact Kaehler manifold, and Z a complex submanifold of codimension $c$. Show that the Poincaré dual of [Z] lies in $\mathrm{H}^{c,c}(\mathrm{X})$.

*Exercise* 16 (H.-C. Wang). Let X be a compact Kaehler manifold. Then $\mathrm{T}_\mathrm{X}$ is trivial if and only if X is a torus. [Show that the Albanese map is an etale covering.]

*Exercise* 17. Let X be a compact connected Kaehler manifold with vanishing Ricci curvature.
(i) If $\omega$ is a holomorphic $p$-form, then $\nabla\omega = 0$. [Compute $\Delta_d\omega = \nabla^*\nabla\omega$. Notice that $\Delta_d\omega = 0$, and integrate over X to conclude.]
(ii) Let $x \in \mathrm{X}$. Use (i) to deduce that the map $\mathrm{H}^0(\mathrm{X}, \Omega^p) \to \left(\wedge^p(T_p^{1,0}\mathrm{X})^\vee\right)^{\mathrm{Hol}_x(\mathrm{X})}$ given by $\omega \mapsto \omega(x)$ is an isomorphism.
(iii) Assume that $\mathrm{Hol}_x(\mathrm{X}) = \mathrm{SU}(\dim \mathrm{X})$. Show that $\mathrm{H}^0(\mathrm{X}, \Omega^p) = 0$ for $0 < p < \dim \mathrm{X}$. [Use (i), and show that the representation $\wedge^p\sigma^\vee$ ($\sigma$ the standard representation of $\mathrm{SU}(\dim \mathrm{X})$) is irreducible.]

**Geometric Invariant Theory.**

*Exercise* 18. We consider the action of $\mathbf{C}^*$ on $\mathbf{C}^4$ with weight $(1, 1, -1, -1)$.
(i) Show that the algebra of invariants can be identified with

$$\mathrm{A}_0 = \mathbf{C}[\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{W}]/(\mathrm{XW} - \mathrm{YZ}).$$

(ii) To form a GIT quotient, one also needs a linearisation. In our case this is nothing but a $\mathbf{Z}$-grading on A[Q], where A is the polynomial ring $\mathbf{C}[\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \mathrm{W}]$ with $\mathbf{Z}$-grading corresponding to the action of $\mathbf{C}^*$ (i.e., $\mathrm{X}, \mathrm{Y} \in \mathrm{A}_1$, $\mathrm{Z}, \mathrm{W} \in \mathrm{A}_{-1}$, and $\mathrm{A}_0$ is as in (i)); the GIT quotient is then $\mathrm{Proj}(\mathrm{A[Q]}_0)$. Consider the three gradings on A[Q] determined by $\mathrm{Q} \in \mathrm{A[Q]}_{-1}$, $\mathrm{Q} \in \mathrm{A[Q]}_0$, $\mathrm{Q} \in \mathrm{A[Q]}_1$, and denote by $\mathrm{X}_-, \mathrm{X}_0, \mathrm{X}_+$ the corresponding GIT quotients. Identify $\mathrm{X}_0$ with $\mathrm{Spec}(\mathrm{A}_0)$, and $\mathrm{X}_-$ (resp. $\mathrm{X}_+$) with the blow up of $\mathrm{X}_0$ along along (X, Z) (resp. (Y, W)). The induced rational map $X^- \to X^+$ is the *Atiyah flop*.

**Equivariant Cohomology.**

*Exercise* 19. Let $\mathrm{G} = \mathrm{Gr}(2, \mathrm{V})$ be the Grassmannian of lines in $\mathbf{P}^3 = \mathbf{P}(\mathrm{V})$, with tautological bundles $\mathscr{S}$ and $\mathscr{Q}$. The torus $\mathrm{T} = (\mathbf{C}^*)^4$ acts on $\mathbf{P}^3$ by

$$(t_0, t_1, t_2, t_3)[x_0 : x_1 : x_2 : x_3] = [t_0^{-1}x_0 : t_1^{-1}x_1 : t_2^{-1}x_2 : t_3^{-1}x_3].$$

(i) Show that there is an induced action of T on G, and that the fixed locus $\mathrm{G}^\mathrm{T}$ consists of the 6 lines $\mathrm{L}_\lambda$ (where $\lambda = (\lambda_1, \lambda_2)$ satisfies $0 \le \lambda_1 < \lambda_2 \le 3$) given by $x_i = 0$, $i \ne \lambda_1, \lambda_2$. For each $\lambda$ compute the T-equivariant Chern classes of the T-equivariant vector bundles $\mathscr{S}_{\mathrm{L}_\lambda} = \mathrm{L}_\lambda$ and $\mathrm{N}_{\mathrm{L}_\lambda/\mathrm{G}} = \mathrm{T}_{\mathrm{G},\mathrm{L}_\lambda} = \mathrm{Hom}(\mathrm{L}_\lambda, \mathrm{V}/\mathrm{L}_\lambda)$ over $\mathrm{Spec}(\mathbf{C})$. (These are nothing but linear representations of T; the T-equivariant Chern classes are elements of $\mathrm{H}^*_\mathrm{T}(*; \mathbf{Q}) \simeq \mathrm{Sym}^*(\mathrm{T}^\vee \otimes \mathbf{Q})$, where $\mathrm{T}^\vee$ is the group of characters of T.)
(ii) Use (i) and the Atiyah-Bott integration formula to compute

$$\chi(\mathrm{G}) = \int_\mathrm{G} c_4(\mathrm{T}_\mathrm{G}).$$

(iii) Use (i) and the Atiyah-Bott integration formula to compute

$$\int_{\mathrm{G}} c_1(\mathscr{S})^4 = 2,$$

and give a geometric interpretation of the result. [For the interpretation make use of the definition of Chern classes via degeneracy loci; it is convenient to consider $c_1(\mathscr{Q}) = c_1(\mathscr{S})$, since $\mathrm{H}^0(\mathrm{G}, \mathscr{S}) = 0$ and $\mathrm{H}^0(\mathrm{G}, \mathscr{Q}) = \mathrm{V}$.]

(iv) Use and the Atiyah-Bott integration formula to compute

$$\int_{\mathrm{G}} c_4(\mathrm{Sym}^3(\mathscr{S}^\vee)) = 27$$

and give a geometric interpretation of the result.

**Deformation Theory.**

*Exercise* 20. Let $\mathscr{A}$ be the category of Artin local $\mathbf{C}$-algebras with $\mathbf{C}$.

(i) Let X be an algebraic scheme, and $x \in \mathrm{X}$ a closed point. Consider the functor $F = h_{\mathrm{X},x} : \mathscr{A} \to \mathbf{Set}$ given by letting $F(\mathrm{A})$ be the set of morphisms of schemes $f : \mathrm{Spec}(\mathrm{A}) \to \mathrm{X}$ whose underlying map of spaces takes $\mathrm{Spec}(\mathrm{A})$ to $\{x\}$. Show that $F$ is functorially isomorphic to $\mathrm{Hom}(\hat{\mathscr{O}}_{\mathrm{X},x}, -)$.

(ii) Take $\mathrm{X} = \mathrm{Spec}(\mathbf{C}[\mathrm{U}, \mathrm{V}]/(\mathrm{UV}))$, $x = (\mathrm{U}, \mathrm{V})$. Show that $t_F = F(\mathbf{C}[\mathrm{T}]/(\mathrm{T}^2))$ is a $\mathbf{C}$-vector space of dimension 2, with basis $e, f$ given by $e(\mathrm{U}) = \mathrm{T}, e(\mathrm{V}) = 0$, $f(\mathrm{U}) = 0$, $f(\mathrm{V}) = \mathrm{T}$.

(iii) Show that an element $v = ae + bf \in t_F$ lifts to a morphism

$$V : \mathbf{C}[[\mathrm{U}, \mathrm{V}]]/(\mathrm{UV}) \to \mathbf{C}[\mathrm{T}]/(\mathrm{T}^3)$$

if and only if $a = 0$ or $b = 0$.